THE APPLICATION OF GAUSSIAN CHANNEL THEORY TO THE ESTIMATION OF INFORMATION TRANSMISSION RATES IN NEURAL POPULATION

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ABSTRACT

We study information transmission properties of the nonlinear model of the neuronal system. The model is a summing network of finite number N FitzHugh-Nagamo equations [3],

$$\begin{aligned} \epsilon \dot{v}_i &= -v_i\left(v_i^2 - \frac{1}{4}\right) - w_i + A - b + S(t) + \xi_i(t), \\ \dot{w}_i &= v_i - w_i, \end{aligned} \tag{1}$$

driven by a stationary stochastic signal $S(t)$ and independent white noise $\xi_i(t)$, where $i = 1..N$. The v_i are fast variables (membrane voltages) and w_i are slow (recovery) variables. The output of the different neurons were added together to give the summed response $\hat{r} = \sum_{i=1}^{N} \theta(v_i)$, where

$$\theta(v_i) = \begin{cases} v_i, & v_i > 0; \\ 0, & v_i < 0. \end{cases}$$

The firing rate, $r(t)$, was obtained by passing the summed response through a filter,

$$\dot{r} = -\frac{1}{\tau_p} r + \hat{r}(t). \tag{2}$$

Assuming the signal $S(t)$ is band-limited, by transforming the stochastic processes $S(t)$, the signal, and $r(t)$, the response, into multidimensional random variables $\vec{X}$ and $\vec{Y}$ [2], we find that the Gaussian channel approximation holds (i.e. the probability density functions $P(\vec{X})$, $P(\vec{Y})$ and $P(\vec{X}, \vec{Y})$ are approximated by Gaussian distributions), enabling us to estimate the mutual information I and the information rate R very well.

The n-dimensional vectors $\vec{X} = [X_1, X_2, .., X_n]$ and $\vec{Y} = [Y_1, Y_2, .., Y_n]$ comprise the elements $X_j = S(t_j)$ and $Y_j = r(t_j)$, where $j = 1..n$ and $t_{j+1} = t_j + \delta t$, where $\delta t < 1/(2f_b)$, where f_b is the cutoff frequency of the signal (or response) power spectrum.

The mutual information I_n for n-dimensional multivariates can be defined as

$$I_n = H(\vec{X}) + H(\vec{Y}) - H(\vec{X}, \vec{Y}), \tag{3}$$

where the entropies are

$$\begin{aligned} H(\vec{X}) &= -\int_{\vec{X}} d\vec{X} P(\vec{X}) \log_2 P(\vec{X}), \\ H(\vec{Y}) &= -\int_{\vec{Y}} d\vec{Y} P(\vec{Y}) \log_2 P(\vec{Y}), \\ H(\vec{X}, \vec{Y}) &= -\int_{\vec{X}} \int_{\vec{Y}} d\vec{X} d\vec{Y} P(\vec{X}, \vec{Y}) \log_2 P(\vec{X}, \vec{Y}). \end{aligned} \tag{4}$$

It is easy to see using expressions (3) and (4) that it is not possible to calculate the information for large n via numerical methods because to find the joint probability densities we need a very

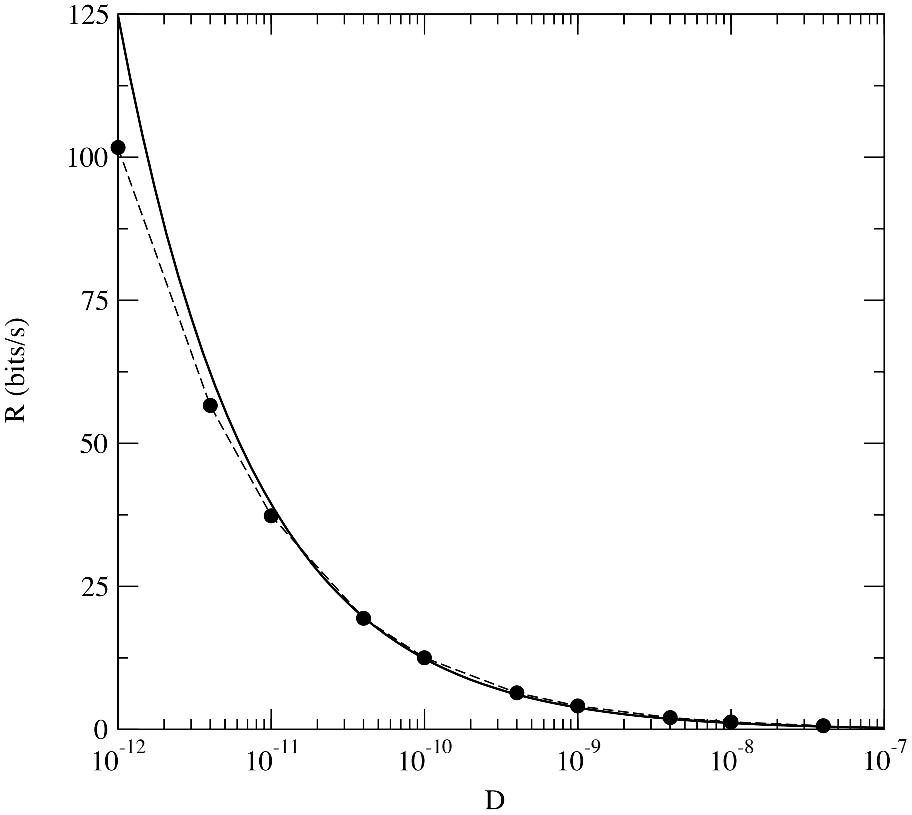

Figure 1: Test results for linear system, $\dot{r} = -r/\tau_p + S(t) + \xi(t)$, $\langle \xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$, $\tau_p = 2.0$. $S(t)$ is a stochastic Gaussian process with correlation function $\langle S(t)S(t+\tau)\rangle = \sigma \exp(-|\tau|/\tau_c)$, signal intensity $\sigma = 1.5 \times 10^{-7}$, and correlation time $\tau_c = 5.0$. The solid line shows the exact analytical result, $R = (\sqrt{1+\tau_c\sigma/D} - 1)/(2\tau_c \log 2)$. The black dots show the results obtained by using the matrix method. The dimensions of $\vec{X}$ and $\vec{Y}$ are $n = 1024$, and the time interval $\delta t = 0.005$.

large amount of data. The Gaussian channel theory [2] reduces the amount of data necessary and enables us to calculate the information. In the Gaussian channel approximation the mutual information can be found as

$$I_n = \frac{1}{2}\log_2 \frac{|\Sigma_{xx}|\,|\Sigma_{yy}|}{\begin{vmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{vmatrix}}. \tag{5}$$

where the elements of the matrices Σ_{xx}, Σ_{yy}, Σ_{xy}, and Σ_{yx} are

$$\begin{aligned} (\Sigma_{xx})_{km} &= \langle X_k X_m\rangle - \langle X_k\rangle\langle X_m\rangle, \\ (\Sigma_{yy})_{km} &= \langle Y_k Y_m\rangle - \langle Y_k\rangle\langle Y_m\rangle, \\ (\Sigma_{xy})_{km} &= \langle X_k Y_m\rangle - \langle X_k\rangle\langle Y_m\rangle, \\ (\Sigma_{yx})_{km} &= \langle Y_k X_m\rangle - \langle Y_k\rangle\langle X_m\rangle, \end{aligned} \tag{6}$$

for $k = 1..n$, $m = 1..n$ [1].

The information rate is

$$R = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{I_n}{n\delta t}. \tag{7}$$

The problem of the information rate R calculation in the limit of infinite-dimensional variables is discussed and a method of dimension reduction [2] is shown using the expression (5). The rate R_n is plotted as a function of n and a limit is sought for large values of n. However, the information value I_n depends on the statistical quality of the calculation of the matrix elements in expression (6). The result of that is divergence of the value of the rate R_n for large n and the nonexistence of the limit (7). We propose a new method for estimation of the information rate R: n is found

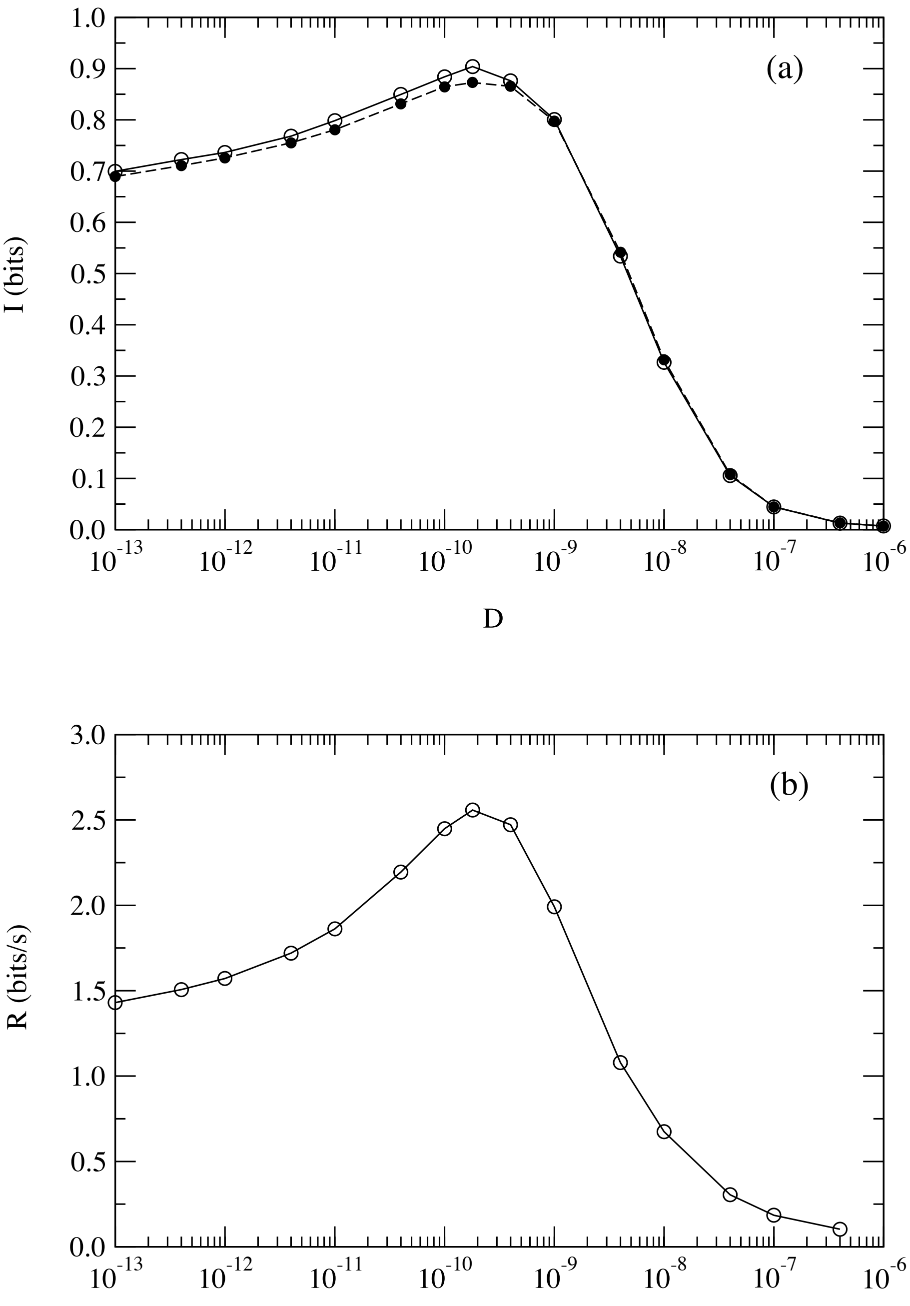

Figure 2: Results for the neuronal model. The number of the neurons is 16. $S(t)$ is a stochastic Gaussian process with correlation function $\langle S(t)S(t+\tau)\rangle = \sigma \exp(-|\tau|/\tau_c)$, signal intensity $\sigma = 1.5 \times 10^{-7}$, and correlation time $\tau_c = 5.0$. In figure (a) the white circles correspond to the results obtained by using expression (3) for unidimensional variables, i.e. $n = 1$. In that case the Gaussian approximation gives $I = I_1 = -\frac{1}{2}\log_2(1-c^2)$, where the cross-correlation coefficient is $c = \langle (S-\langle S\rangle)(r-\langle r\rangle)\rangle / \sqrt{(\langle S^2\rangle - \langle S\rangle^2)(\langle r^2\rangle - \langle r\rangle^2)}$. The approximated mutual information is shown by black dots. In figure (b) the information rate R is presented as a function of the noise intensity D. The dimension of the variables is $n = 1024$, and $\delta t = 0.005$.

for which $R_n = f(n)$ is a minimum, i.e. $R_n \leq R_m \; \forall m \neq n$, and this minimum becomes the estimation of the rate. The method was tested on a linear system for which an exact analytical solution can be obtained (see figure 1) and applied to the estimation of the information rate for the model (1).

The analysis of the numerical results shows (see figure 2) that the mutual information and the information rate are maximised for identical noise intensities. This phenomena can be applied to the solution of the optimal tuning problem for transmission channels.

Keywords: Information theory, Gaussian channel, Neural coding.

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